It is generally preferable to compute them for the values, ... $a - \frac{1}{2}h$, $a + \frac{1}{2}h$, $a + \frac{3}{2}h$, ... Formulæ for this purpose can be obtained by the simple consideration, that in the scheme on p. 141, it is allowable to treat the odd orders of differences as if they were even, and the even as if they were odd.

In this way all the quantities obtained will correspond to the middle of the intervals of the former supposition. Thus, calling D^{-1} and D^{-2} in this case $D_{\frac{1}{2}}^{-1}$ and $D_{\frac{1}{2}}^{-2}$, it is evident we must have

$$D_{\frac{1}{2}}^{-1} = h\left(\int\frac{d\Delta}{\sqrt{(1+\frac{1}{4}\Delta^2)}}\right)^{-1},$$

$$D_{\frac{1}{2}}^{-2} = \frac{h^2}{\sqrt{(1+\frac{1}{4}\Delta^2)}}\left(\int\frac{d\Delta}{\sqrt{(1+\frac{1}{4}\Delta^2)}}\right)^{-2},$$

or, expanded in powers of Δ,

$$D_{\frac{1}{2}}^{-1} = h\left(\Delta^{-1} + \frac{1}{24}\Delta - \frac{17}{5760}\Delta^3 + \frac{367}{967680}\Delta^5 - \frac{27859}{464486400}\Delta^7 + \dots\right),$$

$$D_{\frac{1}{2}}^{-2} = h^2\left(\Delta^{-2} - \frac{1}{24}\Delta^0 + \frac{17}{1920}\Delta^2 - \frac{367}{193536}\Delta^4 + \frac{27859}{66355200}\Delta^6 - \dots\right).$$

The differences of the first formulæ, although they are of odd orders, are to be taken as equivalent to the simple numbers standing in the scheme on p. 141; while the differences of the second, although of even orders, are all the averages of two adjacent numbers of the same scheme

It is plain we have

$$D_{\frac{1}{2}}^{-2} = -h\,\frac{d\Delta_{\frac{1}{2}}^{-1}}{d\Delta}.$$

ON THE DISTRIBUTION OF PRIMES.

BY PROF. W. W. JOHNSON, ST. JOHN'S COLLEGE, ANNAPOLIS, MARYLAND.

I received a short time sinee a letter from Mr. J. W. L. Glaisher of Trinity College, Cambridge, Eng., who is engaged in reporting upon the subject of Mathematical Tables to the British Association, a short paper in which he gvies a compareson of values of the integral

$$\int_x^{x'}\frac{dx}{\log x}$$

with the actual number of primes counted between x and x'. The tables

employed are Tables of Least Divisors; Burckhardt's 1st three millions published 1814–1817, and Dase's 7th, 8th and 9th million, 1862–1865. The gap of 3 millions and the 10th million are said to exist in manuscript but have not yet been published. The primes are of course indicated in these tables by the absence of a divisor, but no separate list of primes among high numbers has been published.

Mr. Glaisher states that he has had them counted for the whole published 6 millions, and the paper gives the comparison with the formula for every interval of 50,000 in the second and also in the ninth million; the count in these cases having been executed in duplicate. There is but one actual discrepancy between these results and those given by Mr. Hill, as far as they admit of comparison. According to Mr. Glaisher the number of primes in the 3rd half-million is 35,649, error of formula —12; by Mr. Hill's table it is 35,611, error of formula +24.6. Mr. Glaisher's table however gives a much better showing for the formula than Mr. Hill's; for according to the latter there is a cumulative positive error throughout the first three millions, while according to the former the errors in the second million, (though indeed for small interval as proportionately large and and irregular as Mr.Hill points out,) show a marked tendency to balance one another.

Thus to select an extreme case, errors of — 59 and + 37 occur in adjacent 50 thousands; while however the whole error for the 2nd million is only + 9 and that for the 9th million is — 84. Mr. Glaisher states that the numbers of primes counted, for the 1st and 3rd millions, differ widely from those published by Hargrave in 1854, (who makes the descrepancies mueh greater than that given above for the 2nd million,) and he purposes soon to publish the result, for the other millions, together with a comparison with Legendre's formula.

The general correspondenc of the two formulæ may be seen thus: the differential of

$$\frac{x}{A\, log\, x - B} \quad \text{is} \quad \frac{A\, log\, x - B - A}{(A\, log\, x - B)^2}\, dx.$$

When $log\, x$ is large, and A and B near to unity, we have approximately

$$\frac{log\, x - 2}{(log\, x)^2 - 2 log\, x}\, dx = \frac{dx}{log\, x}$$

the differential of the integral formula. It is therefore to be expected that the formula

$$\frac{x}{A\, log\, x - B}$$

with values of A and B empirically determined so as to give the best possible

results within the limits of comparison selected, should within those limits give better results than

$$\int_0 \frac{dx}{logx},$$

notwithstanding the fact that the later is the true formula when the value of x is unlimited.

Prof. Hendrickson U S. Naval Academy proposes the following method of obtainiug the equation of the tangent in terms of its direction ratio, the equation of the curve being in the form $y = f(x)$ . . . . (1)

Let $$m = \frac{dy}{dx} = f'(x) \quad . \; . \; . \; . \; . \; (2)$$

Then x and y being implicitly functions of m, let

$$y - mx = \psi(m) \quad . \; . \; . \; . \; . \; (3)$$

The form of the function $\psi(m)$ is reqnired. Differentiate (3), and we have $$dy - mdx - xdm = \psi'(m)dm.$$

Since $dy = mdx$ and dm is not zero, [(1) representing a curve]

$$-x = \psi'(m).$$

Considering x as a function of m, and integrating

$$\psi(m) = C - \int xpm.$$

To find $\psi(m)$ from (4) it is necessary to express x in terms of m from (2), perform an integration and determine C. In many cases it is easy to determine $\psi(m)$ directly from (3), which requires us to express x in terms of m as before and also to express y in terms of m by elimination from (1) and (2). when on the other hand this elimination is inconvenient, $\psi(m)$ may often be determined by (4).

The equation of the tangent is then $\quad y = mx + \psi(m)$.

EXAPLES.

1. Given $y = log\, x$, then $m = \frac{1}{x}$, whence $x = \frac{1}{m}$ and $y = -\, logm$.

Henc from (3) $-\, log\, m - 1 = \psi(m)$; and the tangeat is $y = mx - (1 + logm)$.

2. Given $y = (x^2 + 1)(x + 1)$, then $m = 3x^2 + 2x + 1$, and

$$x = -\tfrac{1}{3} \pm \surd(\tfrac{1}{3}m - \tfrac{2}{9}). \; . \; . \; (a)$$

Hence from (4) $\psi(m) = C + \frac{1}{3}m \mp 2\surd(\frac{1}{3}m - \frac{2}{9})^3$; therefore

$$y = mx + C + \tfrac{1}{3}m \mp 2\surd(\tfrac{1}{3}m - \tfrac{2}{9})^3.$$

To determine C, let $x = -1$ (say), then $y = 0$, $m = 2$; substituting' [using the lower sign because, if $m = 2$ in (a), the upper sigp gives $x = \frac{1}{3}$,]

$0 = -2 + C + \frac{2}{3} + \frac{16}{27}$ or $C = \frac{20}{27}$; therefore che tangentsared

$y = mx + \frac{20}{27} + \frac{1}{3}m \mp 2\surd(\frac{1}{3}m - \frac{2}{9})^3$.

[Prof. Johnson says that his intention was, in proposing problem 33, to require an integral equation between x and y referred to rectangular axes. The special interest, he remarks, in the problem consists in the avoidance of radicals which have not properly the double sign; and he requests us to propose the problem of finding the rectangular coordinates of the double point not on the axis of x, referring to Mr. Stille's figure in No. 9. Ed.]

FOLIATE CURVES.

BY PROF. E. W, HYDE, ITHACA, N. Y.

Prop. The foliate curves represented by the equatiou $\rho = a \cos n\theta$ (or $\rho = a \sin n\theta$) are hypotrochoids if n* be an integer, and both hypotrochoids and epitroehoids if n be fractional.

1. The equations of the hypotrochoid are

$$(1) \qquad x = (r_1 - r_2)\cos\phi + mr_2 \cos\left(\frac{r_1 - r_2}{r_2}.\phi\right)$$

$$(2) \qquad y = (r_1 - r_2)\sin\phi - mr_2 \sin\left(\frac{r_1 - r_2}{r_2}.\phi\right),$$

in which r_1 = radius of fixed circle,
r_2 = " " rolling " ,
mr_2 = distance of generating point from center of rolling circle, and ϕ = angle between the axis of x and the radius of the rolling circle containing the generating point.

Let $\qquad r_1 = pr_2 = \dfrac{pa}{2(p-1)}$, and $m = p - 1$.

Substituting in (1) and (2) we have

$$(3) \qquad x = \tfrac{1}{2}a[\cos\phi + \cos(p-1).\phi] = a\cos(\tfrac{1}{2}p.\phi)\cos[\tfrac{1}{2}(2-p).\phi],$$

$$(4) \qquad y = \tfrac{1}{2}a[\sin\phi - \sin(p-1).\phi] = a\cos(\tfrac{1}{2}p.\phi)\sin[\tfrac{1}{2}(2-p).\phi].$$

Divide (4) by (3) and we get

$$(5) \qquad \frac{y}{x} = \frac{\sin[\frac{1}{2}(2-p).\phi]}{\cos[\frac{1}{2}(2-p).\phi]} = \tan[\tfrac{1}{2}(2-p).\phi] = \tan\theta,$$ where θ equals the angle between ρ and the axis of x.

$\therefore\ \tfrac{1}{2}(2-p).\phi = \theta$, whence $\phi = \dfrac{2\theta}{2-p}$ and $\tfrac{1}{2}p\phi = \dfrac{p\theta}{2-p}$.

Substituting these values of ϕ in equation (3)

$$x = \rho\cos\theta = a\cos\frac{p\theta}{2-p}.\cos\theta,$$

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* Since n, m, and p are merely numerical muitipliers they are intrinsically positive.